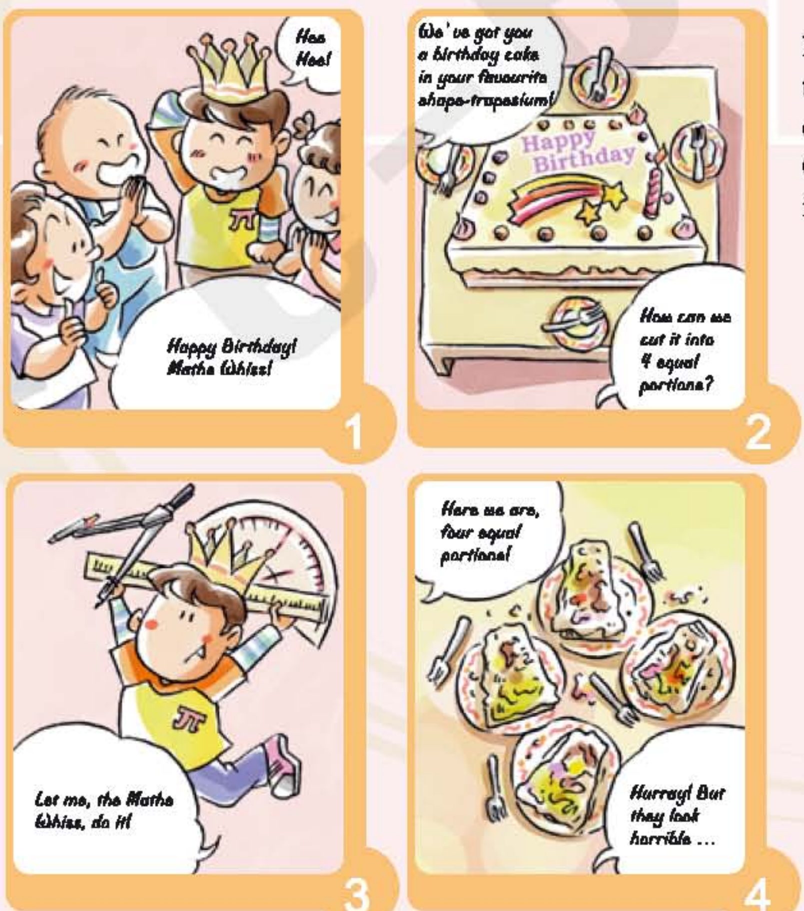
Chapter 12

Polygons

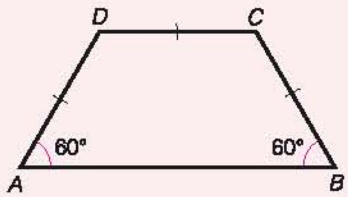
Learning Objectives

After completing this chapter, you will be able to

- understand the properties of the sum of interior angles of a polygon.
- understand the properties of the sum of exterior angles of a convex polygon.
- recognize regular polygons which can tessellate.
- familiarize with the techniques of simple construction by using straight edges and compasses.
- appreciate the method of constructing regular polygons by using straight edges and compasses.



In the figure, $ABCD$ is a trapezium with base angles equal to $60°$ and $AD = BC = DC$. Can you divide this trapezium into four identical trapeziums?



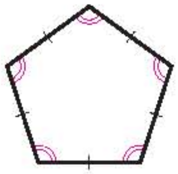
Preview

[Basic knowledge required for this chapter.]

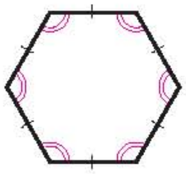
Basic Knowledge

1. A regular polygon is a polygon with equal sides and equal interior angles.

e.g.

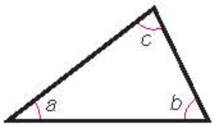


Regular pentagon



Regular hexagon

2. The sum of all interior angles of a triangle is 180°.

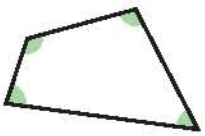


i.e. $a+b+c=180°$

[**Abbreviation:** $\angle$ sum of Δ]

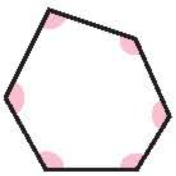
12.1 Sum of Interior Angles of a Polygon

We have learned that the sum of interior angles of a triangle is 180°. So, what is the sum of interior angles of a polygon?



Interior angles of a quadrilateral

Interior angles of a pentagon



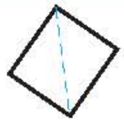
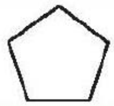
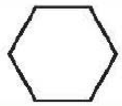
Interior angles of a hexagon

Figure 12.1

Class Activity 12.1

Aim: To explore the sum of interior angles of a polygon

1. Complete the following table. Show Ans.

Polygon	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon
Figure					
Minimum number of triangles divided from the polygon	1	2			
Sum of interior angles of the polygon	180°	( ) × 180°	( ) × 180°	( ) × 180°	( ) × 180°

2. Hence, the sum of interior angles of an n-gon is expected to be ________. Show Ans.

From Class Activity 12.1, we discover the following property.

> The sum of all interior angles of an n-gon is $(n-2)\times 180°$.
> [**Abbreviation:** ∠ sum of polygon]

Notes: **(a)** Although we have only discussed convex polygons in Class Activity 12.1, the above formula can also be applied to concave polygons. For example, the sum of interior angles of the concave polygon in Figure 12.2 is $(5-2)\times 180°$.

(b) The above formula can be proved by deduction. Due to the complexity of the process, the proof is not discussed here.

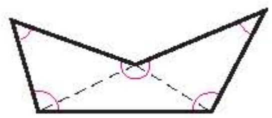
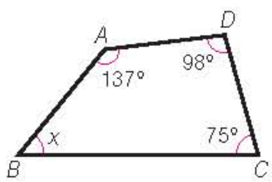


Figure 12.2

Additional Example 12.1

Example 12.1 Finding unknowns through '∠ sum of polygon'

Find x in the figure.



Solution

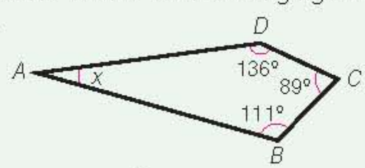
$$x+75°+98°+137°=(4-2)\times 180° \qquad (\angle \text{ sum of polygon})$$
$$x+310°=360°$$
$$x=\underline{\underline{50°}}$$

Classwork 12.1

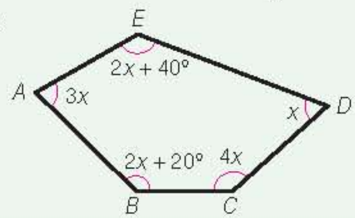
Solution

Find x in each of the following figures.

(a)



(b)



Additional Example 12.2

Example 12.2 Finding an interior angle of a regular polygon through '∠ sum of polygon'

Find the size of an interior angle of a regular 20-gon.

Solution

Sum of interior angles of a regular 20-gon
$= (20 - 2) \times 180°$ (∠ sum of polygon)
$= 3\,240°$

∵ All interior angles of a regular polygon are equal.

∴ An interior angle $= 3\,240° \div 20$
$= \underline{\underline{162°}}$

Classwork 12.2

Solution

Find the size of an interior angle of each of the following regular polygons.

(a) Regular 15-gon
(b) Regular 18-gon

Additional Example 12.3

Example 12.3 Finding number of sides of a polygon through '∠ sum of polygon'

If the sum of interior angles of a polygon is 3 960°, find the number of sides of the polygon.

Solution

Let n be the number of sides of the polygon.

$(n - 2) \times 180° = 3\,960°$ (∠ sum of polygon)
$n - 2 = 22$
$n = 24$

∴ $\underline{\underline{\text{The number of sides of the polygon is 24.}}}$

Classwork 12.3

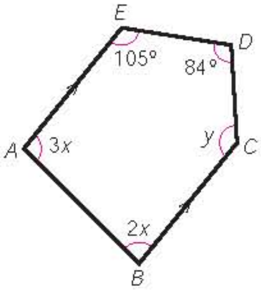
Solution

If the sum of interior angles of a polygon is 4 680°, find the number of sides of the polygon.

Additional Example 12.4

Example 12.4 Finding unknowns through '∠ sum of polygon' and other theorems

Find x and y in the figure.

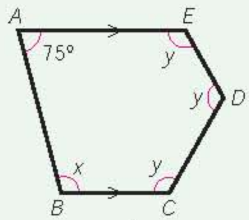


Classwork 12.4

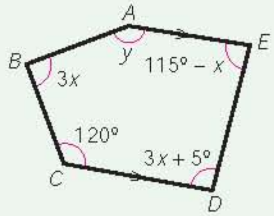
Solution

Find x and y in each of the following figures.

(a)



(b)



Solution

$3x+2x=180°$ (int. ∠s, $AE \,//\, BC$)

$5x=180°$

$x=\underline{\underline{36°}}$

$y+84°+105°+3x+2x=(5-2)\times180°$ (∠ sum of polygon)

$y+84°+105°+180°=540°$

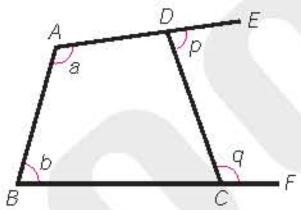
$y+369°=540°$

$y=\underline{\underline{171°}}$

Additional Example 12.5

Example 12.5 Proof for the relation between angles through '∠ sum of polygon'

In the figure, ADE and BCF are straight lines. Prove that $a+b=p+q$.



Proof

$\angle ADC=180°-p$ (adj. ∠s on st. line)

$\angle BCD=180°-q$ (adj. ∠s on st. line)

In quadrilateral $ABCD$,

$a+b+\angle ADC+\angle BCD=(4-2)\times180°$ (∠ sum of polygon)

$a+b+180°-p+180°-q=360°$

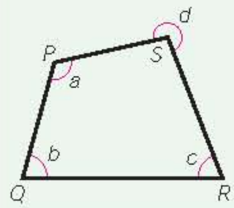
$a+b+360°-(p+q)=360°$

$a+b=360°-360°+(p+q)$

$=p+q$

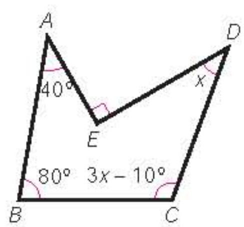
Classwork 12.5 Solution

In the figure, prove that $d=a+b+c$.



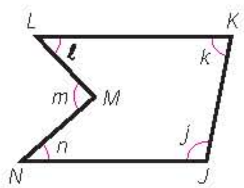
Skills Upgrading Corner 12.1

Solution **1.** Find x in the figure.



Solution **2.** If each interior angle of a regular polygon is 135°, find the number of sides of the regular polygon.

Solution **3.** In the figure, prove that $m+180°=j+k+\ell+n$.



Exercise 12A

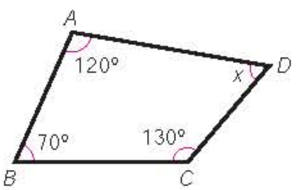
Level 1

1. Find the sum of interior angles of each of the following polygons.

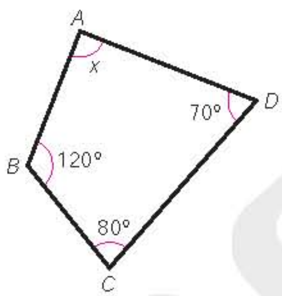
(a) Nonagon **(b)** 12-gon **(c)** 20-gon

Find x in each of the following figures. (**2** – **3**)

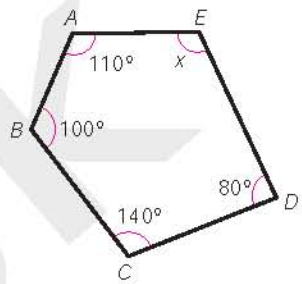
2. (a)



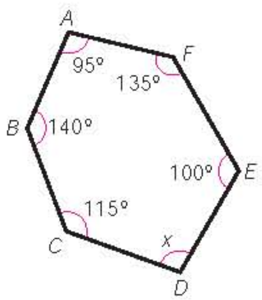
(b)



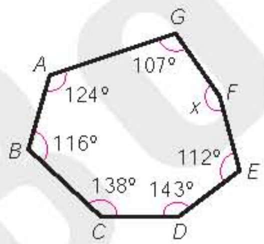
(c)



(d)



(e)



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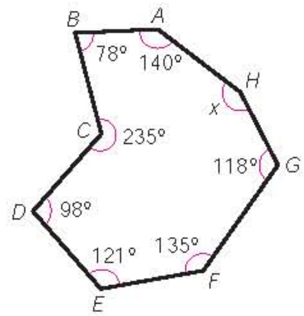
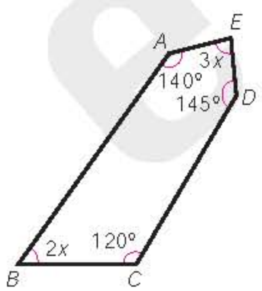


3. (a)



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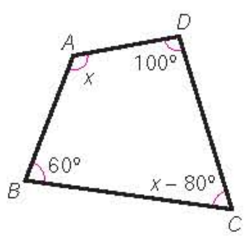
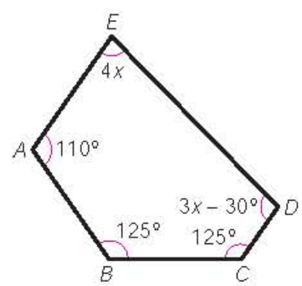


(c)



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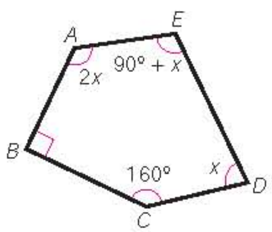
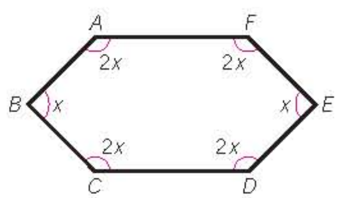


(e)



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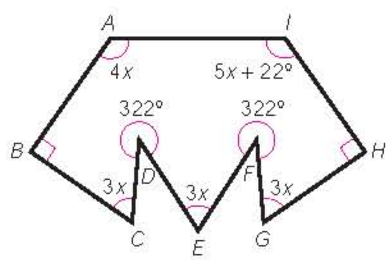


4. Find the size of an interior angle of each of the following regular polygons.
 (a) Regular hexagon **(b)** Regular 16-gon **(c)** Regular 36-gon

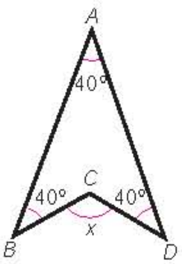
5. If each of the following is the sum of interior angles of a polygon, find the number of sides of the polygon.
 (a) 1 260° **(b)** 4 140° **(c)** 4 500°

6. If each of the following is the size of an interior angle of a regular polygon, find the number of sides of the regular polygon.
 (a) 144° **(b)** 168° **(c)** 171°

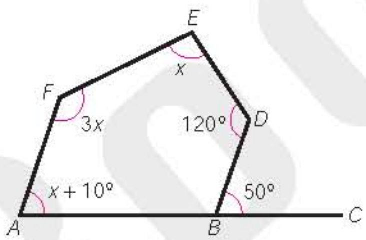
Level 2

7. Find x in each of the following figures.

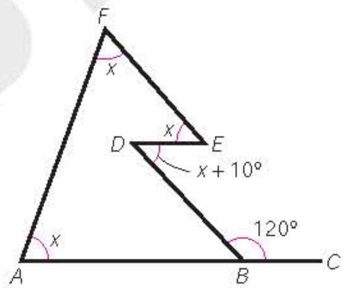
(a)



(b) ABC is a straight line.

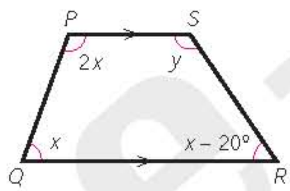


(c) ABC is a straight line.

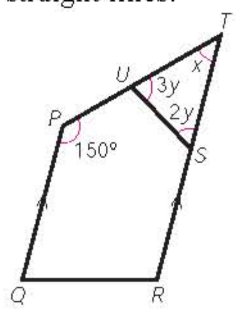


8. Find x and y in each of the following figures.

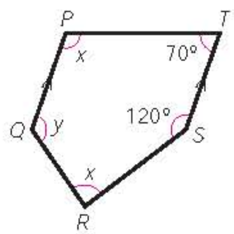
(a)



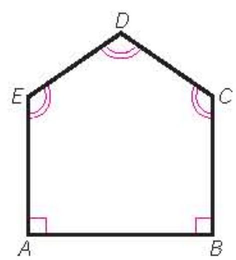
(b) PUT and TSR are straight lines.



(c)



9. The figure shows the side view of a hut. If $\angle A = \angle B = 90°$ and $\angle C = \angle D = \angle E$, find $\angle D$.



10. A hexagon has three equal interior angles, and the sum of the other three interior angles is 450°. Find the size of each of the three equal interior angles.

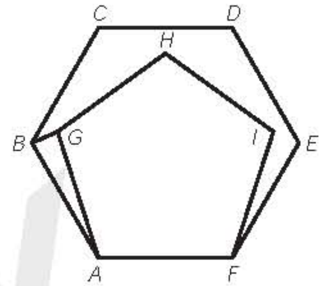
11. The sizes of interior angles of a quadrilateral are in the ratio of $1:2:3:4$. Find the size of the smallest interior angle.

12. If the sum of interior angles of a 13-gon is $360°$ more than that of an n-gon, find the value of n.

13. In the figure, $ABCDEF$ is a regular hexagon and $AGHIF$ is a regular pentagon.

(a) Find $\angle BAG$.

(b) Find $\angle AGB$.

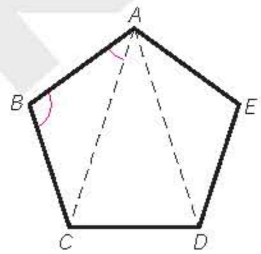


14. In the figure, $ABCDE$ is a regular pentagon.

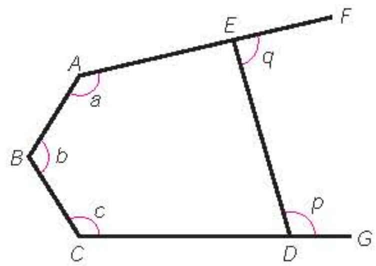
(a) Find $\angle ABC$.

(b) Find $\angle BAC$.

(c) Prove that AC and AD *trisect* $\angle BAE$.
[Hint: i.e. Prove that $\angle BAC = \angle CAD = \angle DAE$.]



15. In the figure, AEF and CDG are straight lines. Prove that $a + b + c = 180° + p + q$.



12.2 Sum of Exterior Angles of a Convex Polygon

By producing each side of a convex polygon, all exterior angles of the polygon can be formed. Figure 12.3 shows the exterior angles formed by producing the sides of a pentagon in different directions.

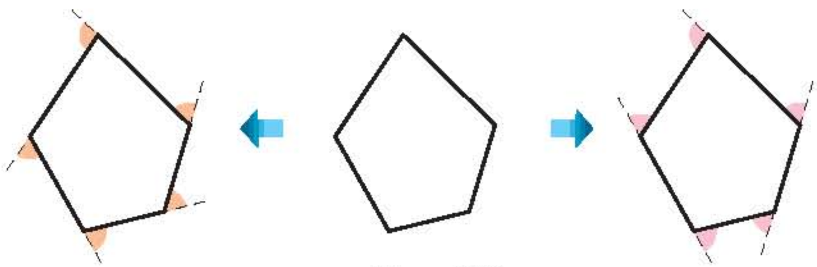


Figure 12.3

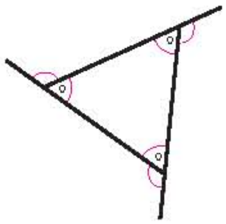
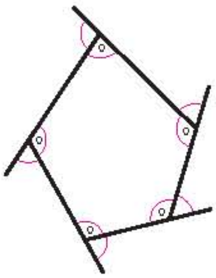
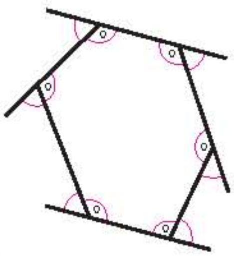
trisect 三等分

Class Activity 12.2

Aim: To explore the sum of exterior angles of a convex polygon

1. Complete the following table.

Show Ans.

	Sum of all angles	Sum of interior angles	Sum of exterior angles
	$\because$ There are ______ pairs of $\therefore$ Sum of all angles = ______ $\times 180°$ = ______	This is a triangle, sum of interior angles= ______ .	Sum of exterior angles = ______ − ______ = ______
	$\because$ There are ______ pairs of $\therefore$ Sum of all angles = ______ $\times 180°$ = ______	This is a ______ , sum of interior angles= ______ .	Sum of exterior angles = ______ − ______ = ______
	$\because$ There are ______ pairs of $\therefore$ Sum of all angles = ______ $\times 180°$ = ______	This is a ______ , sum of interior angles= ______ .	Sum of exterior angles = ______ − ______ = ______

2. Hence, the sum of exterior angles of a convex polygon is expected to be ______________ .

Show Ans.

From Class Activity 12.2, we discover the following property.

> The sum of all exterior angles of a convex polygon is 360°.
> [**Abbreviation:** sum of ext. ∠s of polygon]

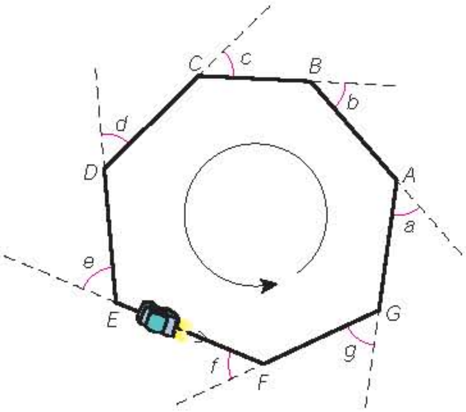
Note: The above property can be proved by deduction. Interested students may try to prove the property using the method mentioned in the above Class Activity.

I.T. Tool

Extension 12.1

Solution

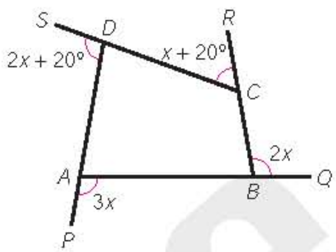
In the figure, a race car completes a lap along a heptagonal race track. What is the sum of the angles turned by the race car?



Additional Example 12.6

Example 12.6 Finding unknowns through 'sum of ext. ∠s of polygon'

In the figure, DAP, ABQ, BCR and CDS are straight lines. Find x.



Solution

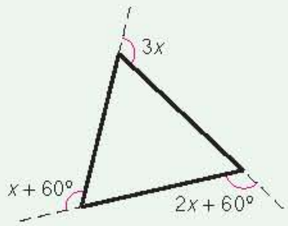
$$3x + 2x + (x + 20°) + (2x + 20°) = 360° \quad \text{(sum of ext. } \angle\text{s of polygon)}$$
$$8x + 40° = 360°$$
$$8x = 320°$$
$$x = \underline{\underline{40°}}$$

Classwork 12.6

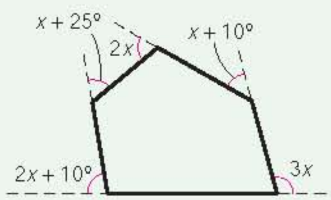
Solution

In each of the following figures, all dotted lines are produced from the solid lines. Find x.

(a)



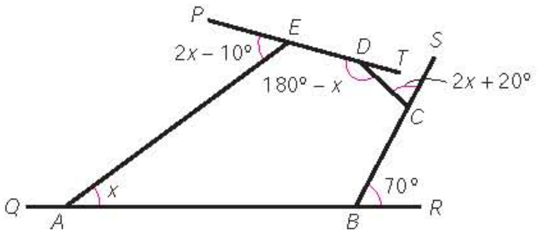
(b)



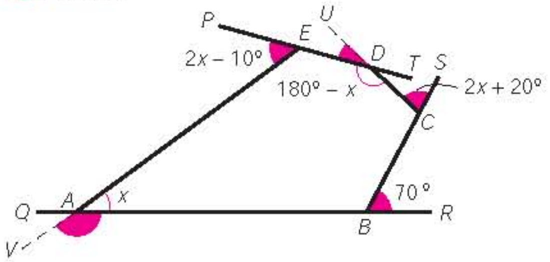
Additional Example 12.7

Example 12.7 Finding unknowns through 'sum of ext. ∠s of polygon'

In the figure, BCS, $PEDT$ and $QABR$ are straight lines. Find x.



Solution



$\angle UDE = 180° - (180° - x)$ (adj. ∠s on st. line)

$= x$

$\angle VAB = 180° - x$ (adj. ∠s on st. line)

$$70° + (2x + 20°) + x + (2x - 10°) + (180° - x) = 360° \quad \text{(sum of ext. ∠s of polygon)}$$

$$4x + 260° = 360°$$

$$4x = 100°$$

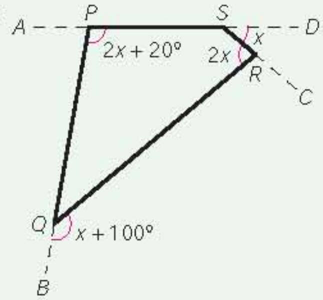
$$x = \underline{\underline{25°}}$$

Classwork 12.7

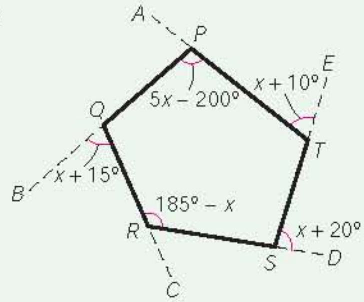
Solution

In each of the following figures, all dotted lines are produced from the sides of the polygon. Find x.

(a)



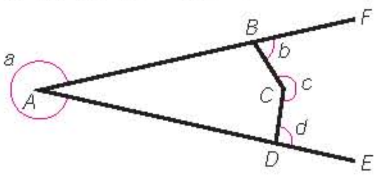
(b)



Additional Example 12.8

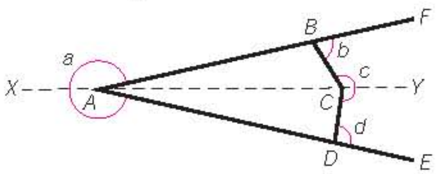
Example 12.8 Proof for the relation among angles through 'sum of ext. ∠s of polygon'

In the figure, ABF and ADE are straight lines. Prove that $a + b + c + d = 720°$.



Proof

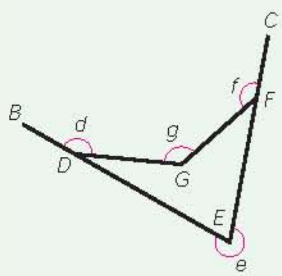
Draw straight line $XACY$.



Classwork 12.8

Solution

In the figure, BDE and CFE are straight lines. Prove that $d + e + f + g = 720°$.



Consider ΔABC,
$\angle BAX + \angle CBF + \angle BCY = 360°$ (sum of ext. $\angle$s of polygon)
Consider ΔACD,
$\angle XAD + \angle CDE + \angle DCY = 360°$ (sum of ext. $\angle$s of polygon)

$$\begin{aligned}\therefore \quad a+b+c+d &= \angle BAX + \angle CBF + \angle BCY + \angle XAD + \angle CDE + \angle DCY \\ &= 360° + 360° \\ &= 720°\end{aligned}$$ (proved)

Additional Example 12.9

Example 12.9 Finding number of sides of a polygon through '$\angle$ sum of polygon' and 'sum of ext. $\angle$s of polygon'

If 6 times an exterior angle of a regular polygon is 117° less than its interior angle, find the number of sides of the regular polygon.

Solution

[Analysis: Set up an equation through '$\angle$ sum of polygon' and 'sum of ext. $\angle$s of polygon'.]

Let n be the number of sides of the regular polygon.

$\therefore$ Each exterior angle $= \dfrac{360°}{n}$ (sum of ext. $\angle$s of polygon)

Each interior angle $= \dfrac{(n-2)\times 180°}{n}$ ($\angle$ sum of polygon)

$$\begin{aligned}\therefore \quad 6\left(\frac{360°}{n}\right) + 117° &= \frac{(n-2)\times 180°}{n} \\ 6\times 360° + 117°n &= 180°(n-2) \\ 2\,160° + 117°n &= 180°n - 360° \\ 2\,520° &= 63°n \\ n &= 40\end{aligned}$$

$\therefore$ The number of sides of the regular polygon is 40.

Classwork 12.9

Solution

If 8 times an exterior angle of a regular polygon is 140° less than its interior angle, find the number of sides of the regular polygon.

Skills Upgrading Corner 12.2

Solution

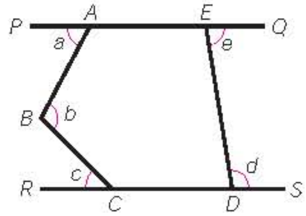
1. If an exterior angle of a regular polygon is 24° less than $\frac{1}{6}$ of its interior angle, find the number of sides of the regular polygon.

Solution

2. Is there a regular polygon with exterior angles of 16° each? Explain briefly.

Solution

3. In the figure, $PAEQ$ and $RCDS$ are straight lines. Prove that $a+c+d+e=180°+b$.



Exercise 12B

Level 1

1. Find the size of an exterior angle of each of the following regular polygons.

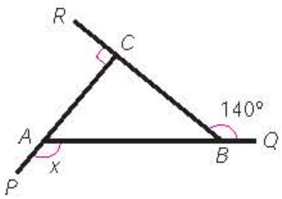
(a) Regular decagon **(b)** Regular 16-gon **(c)** Regular 60-gon

2. If each of the following is the size of an interior angle of a regular polygon, find the number of sides of the regular polygon.

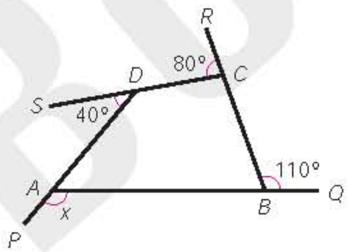
(a) 108° **(b)** 162° **(c)** 165°

In each of the following figures, each marked angle is an exterior angle of the polygon. Find x. (**3** – **4**)

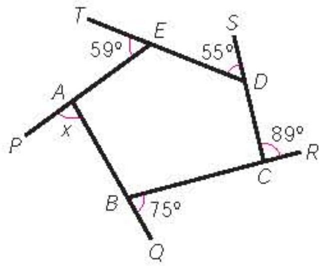
3. (a)



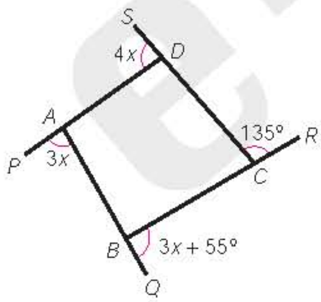
(b)



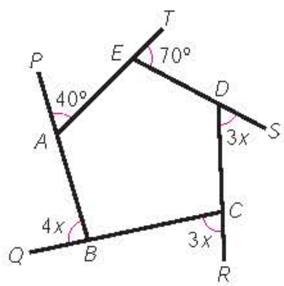
(c)



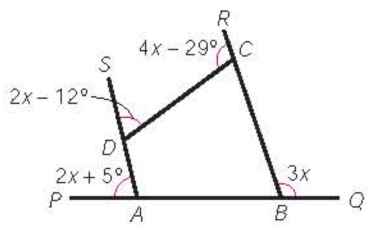
4. (a)



(b)



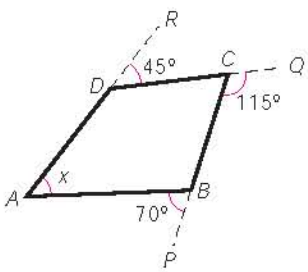
(c)



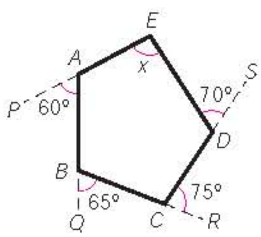
Level 2

In each of the following figures, all dotted lines are produced from the solid lines. Find x. (**5** – **6**)

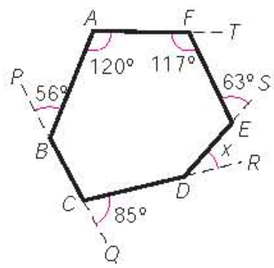
5. (a)



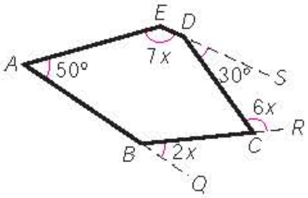
(b)



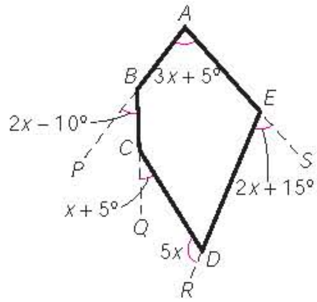
(c)



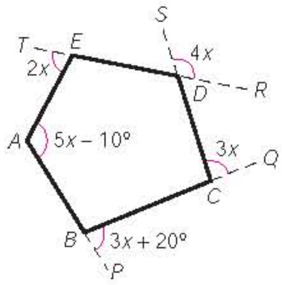
6. **(a)**



(b)



(c)



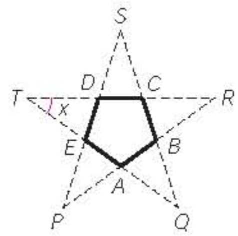
7. If the exterior angles of a convex hexagon are x, $3x$, $5x$, $7x$, $9x$ and $11x$ respectively, find the size of the largest exterior angle.

8. If an interior angle of a regular polygon is 5 times its exterior angle, find the number of sides of the regular polygon.

9. If twice of an interior angle of a regular polygon is 144° more than its exterior angle, find the number of sides of the regular polygon.

10. If twice of an exterior angle of a regular polygon is 140° less than its interior angle, find the number of sides of the regular polygon.

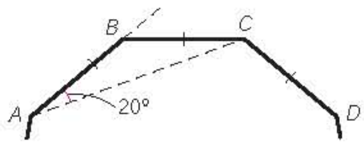
11. The figure shows a regular pentagon $ABCDE$. The dotted lines are produced from the sides of the pentagon to form a star. Find x.



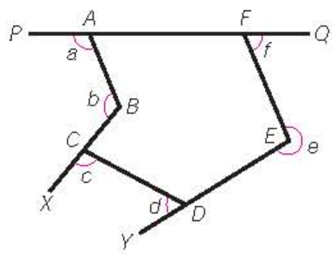
12. In the figure, AB, BC and CD are three sides of a regular polygon where $\angle BAC = 20°$.

(a) Find the size of an exterior angle of the regular polygon.

(b) Find the number of sides of the regular polygon.



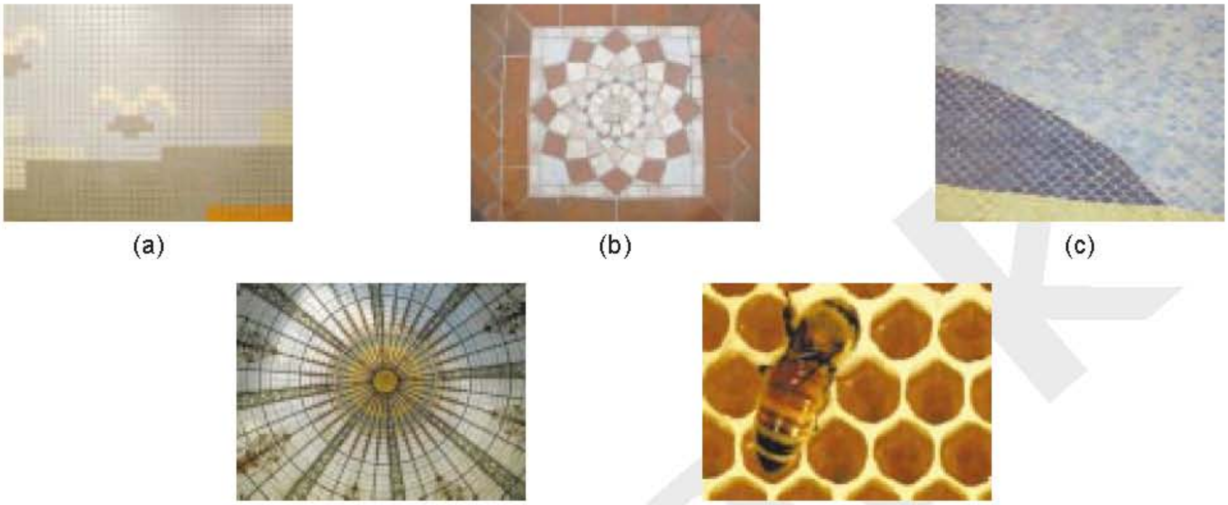
13. In the figure, $PAFQ$, BCX and EDY are straight lines. Prove that $a + b + c + d + e + f = 720°$.



12.3 Tessellation

A Tessellation

It is common to see floors, decorations or things in nature covered with figures in one shape or more around us. The following are some examples.



(a) (b) (c) (d) (e)

Figure 12.4

Repeated use of figures of one kind or more to cover a plane without gaps and without overlaps is called **tessellation**.

e.g. Figures 12.4(a), (c) and (e) show tessellations using one kind of figures, while Figures 12.4(b) and (d) show tessellations using more than one kind of figures.

B Regular tessellation

When one kind of regular polygons in the same size are used in tessellation, we call this regular tessellation. So, how many kinds of regular polygons can tessellate?

Class Activity 12.3

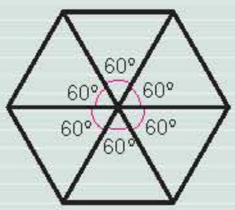
Aim: To explore regular polygons which can tessellate

1. The sum of angles at a point is ____________________ . Show Ans.

2. Can equilateral triangles tessellate? Why? Show Ans.

____

____



tessellation 密鋪

3. Complete the following table. Show Ans.

Number of sides of regular polygon	3	4	5	6	7	8
Interior angle I	60°					
$\dfrac{360°}{I}$						

4. If a regular polygon can tessellate, what characteristic does the value of $\dfrac{360°}{I}$ have? Show Ans.

____

5. Which regular polygons can tessellate? Show Ans.

____

From Class Activity 12.3, if a regular polygon can tessellate, each of its interior angles should be a factor of 360°. Thus only three kinds of regular polygons can tessellate. They are equilateral triangles, squares and regular hexagons. (See Figure 12.5)

(a) (b) (c)

Figure 12.5

Non-foundation Part

12.4 Simple Constructions

Construction refers to the use of a straight edge and a pair of compasses in drawing geometric figures. The straight edge mentioned here is only used for drawing straight lines but not for measuring, therefore the scale is not necessary.

Euclid (around 330BC - 275BC), an ancient Greek mathematician, recorded a number of constructions in his book 'The Elements'. To ensure the constructions were correct, Euclid also provided a deductive proof to support each construction in his book.

Next, we will introduce some simple constructions with proofs.

Archaeologists believe that geometric figures drawn by ancient Greek craftsmen were based on constructions with simple tools such as a rope. When a rope was tightened up, it could be treated as a straight edge. When one end of the rope was fixed at a point, and another end was tied to a movable stone, the whole setting could be treated as a pair of compasses.

archaeologist 考古學家

A Copying an angle

Copying an angle means drawing another angle of the same size. Table 12.1 shows the steps in copying an angle of the same size as $\angle BAC$ on line segment XY.

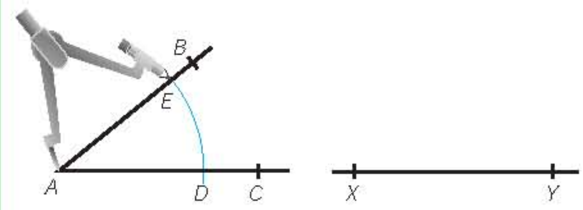
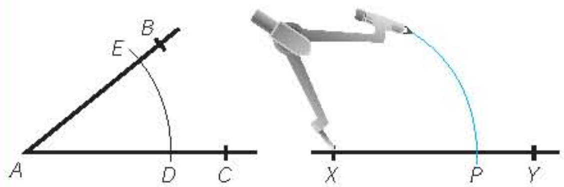
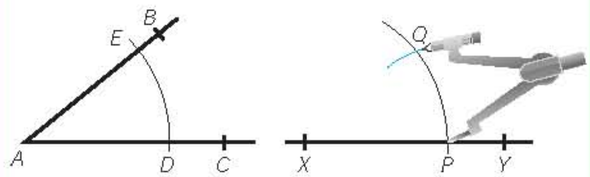
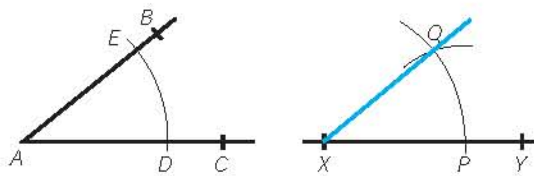
Step 1	With A as the centre, choose an appropriate radius and draw an arc cutting AB and AC at E and D respectively.	
Step 2	With X as the centre and AD as the radius, draw an arc cutting XY at P.	
Step 3	With P as the centre and DE as the radius, draw another arc cutting the arc drawn in step 2 at Q.	
Step 4	Join XQ, we have $\angle QXP = \angle BAC$.	

Table 12.1

The arc drawn should be longer than arc DE.

I.T. Tool

Proof: Join DE and PQ. (See Figure 12.6)

$\because$ $AD = XP$ (by construction)

$AE = XQ$ (by construction)

$DE = PQ$ (by construction)

$\therefore$ $\Delta ADE \cong \Delta XPQ$ (S.S.S.)

$\therefore$ $\angle EAD = \angle QXP$ (corr. $\angle$s, $\cong$ Δs)

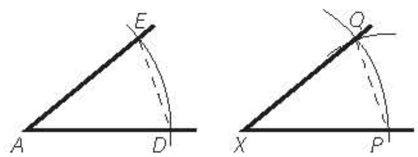


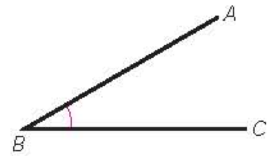
Figure 12.6

Extension 12.2

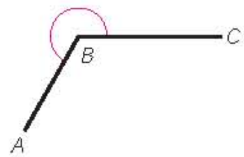
Solution

Copy the following angles with a straight edge and a pair of compasses. Then check your construction with a protractor.

(a)



(b)



B Bisecting an angle

I. Constructing an angle bisector

An **angle bisector** is a straight line which divides an angle into two equal halves. For example, in Figure 12.7, AD is the angle bisector of $\angle BAC$.

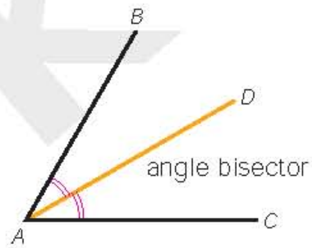


Figure 12.7

Table 12.2 shows the steps in constructing the angle bisector of $\angle BAC$.

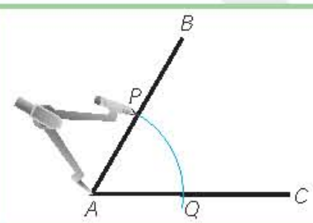
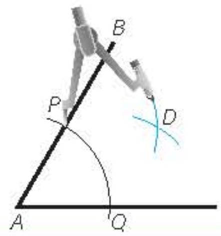
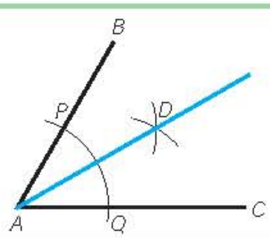
Step 1	With A as the centre, choose an appropriate radius and draw an arc cutting AB and AC at P and Q respectively.	
Step 2	With P and Q as centres, draw two arcs with radii longer than $\frac{1}{2}PQ$, which intersect at D.	
Step 3	Join AD. AD is the angle bisector of $\angle BAC$.	

Table 12.2

I.T. Tool

Proof: Join DP and DQ. (See Figure 12.8)

$\because$ $AP = AQ$ (radii) ◀ Radii of the same circle

$DP = DQ$ (by construction)

$AD = AD$ (common side)

$\therefore$ $\Delta PAD \cong \Delta QAD$ (S.S.S.)

$\therefore$ $\angle PAD = \angle QAD$ (corr. $\angle$s, $\cong$ Δs)

i.e. AD is the angle bisector of $\angle BAC$.

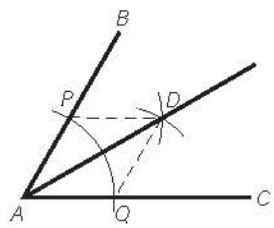


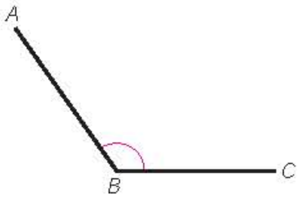
Figure 12.8

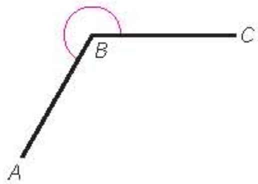
angle bisector 角平分綫

Extension 12.3

Solution

Bisect the following angles with a straight edge and a pair of compasses. Then check your construction with a protractor.

(a) 

(b) 

(c) 

II. Constructing special angles (90°, 60°, 45°, 30°)

Even without a protractor on hand, we can still construct special angles such as 90°, 60°, 45°, 30° etc. with a straight edge and a pair of compasses.

(a) Constructing angles of 90° and 45°

Since $90° = \frac{1}{2} \times 180° = \frac{1}{2}$ straight angle and $45° = \frac{1}{2} \times 90°$, bisecting a straight angle can get an angle of 90°; bisecting an angle of 90° can get an angle of 45°. Table 12.3 shows the steps in constructing angles of 90° and 45°.

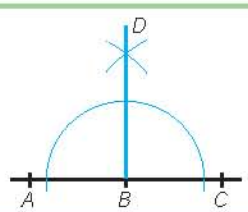
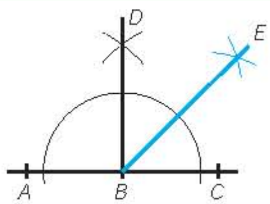
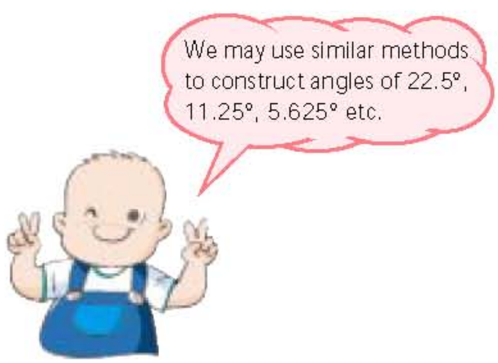
Step 1	Draw a straight angle ABC.	
Step 2	Bisect $\angle ABC$, we have $\angle ABD = \angle CBD = 90°$.	
Step 3	Bisect $\angle CBD$, we have $\angle CBE = \angle DBE = 45°$.	

Table 12.3

I.T. Tool



(b) Constructing an angle of 60°

Since each interior angle of an equilateral triangle is 60°, we may construct an angle of 60° using this property. Table 12.4 shows the steps in constructing an angle of 60°.

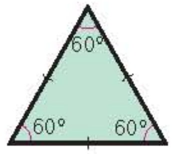


Figure 12.9

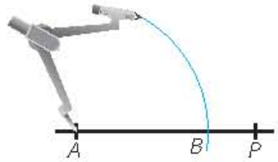
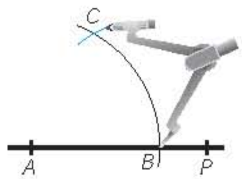
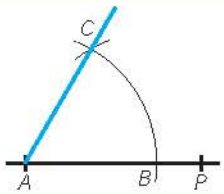
Step 1	Draw a straight line AP.	
Step 2	With A as the centre, choose an appropriate radius and draw an arc cutting AP at B.	
Step 3	With B as the centre and AB as the radius, draw another arc cutting the arc in step 2 at C.	
Step 4	Join AC, we have $\angle CAB = 60°$.	

Table 12.4

I.T. Tool

Proof: Join BC. (See Figure 12.10)

$\because$ $AB = AC$ (radii)

$\therefore$ $\angle ABC = \angle ACB$ (base $\angle$s, isos. Δ)

$\because$ $BC = AB$ (by construction)

$\therefore$ $\angle CAB = \angle ACB$ (base $\angle$s, isos. Δ)

$\therefore$ $\angle CAB = \angle ABC = \angle ACB$

$$\angle CAB + \angle ABC + \angle ACB = 180° \quad (\angle \text{ sum of } \Delta)$$
$$3\angle CAB = 180° \quad (\text{proved})$$
$$\angle CAB = 60°$$

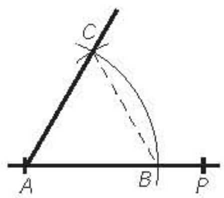


Figure 12.10

Note: In fact, ABC in Figure 12.10 is an equilateral triangle.

Additional Example 12.10

Example 12.10 Constructing specific angles

Construct the following angles with a straight edge and a pair of compasses.

(a) 75°

(b) 150°

Classwork 12.10

Solution

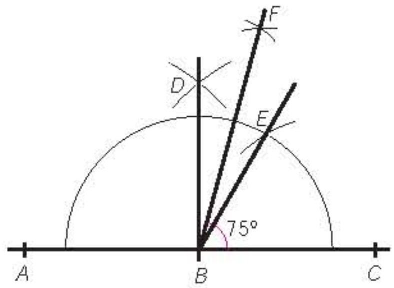
Construct the following angles with a straight edge and a pair of compasses.

(a) 105°

(b) 135°

Solution

(a)

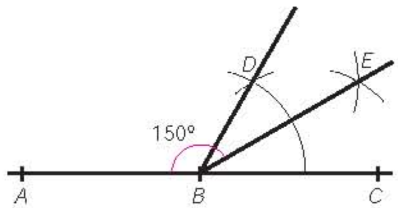


$75° = 60° + 15°$, $15° = \frac{1}{2} \times 30°$

Steps of construction:

1. Construct $\angle DBC = 90°$ and $\angle EBC = 60°$.
2. $\angle DBE = 90° - 60° = 30°$
   Construct the angle bisector BF of $\angle DBE$, we have $\angle FBE = 15°$.
3. $\angle FBC = 60° + 15° = 75°$

(b)



$150° = 180° - 30°$

Steps of construction:

1. Construct $\angle DBC = 60°$.
2. Construct the angle bisector BE of $\angle DBC$, we have $\angle EBC = 30°$.
3. $\angle ABE = 180° - 30° = 150°$

C Constructing a perpendicular bisector

A **perpendicular bisector** is a straight line dividing a line segment into two equal parts and perpendicular to this line segment.

In Figure 12.11, CD is the perpendicular bisector of AB.
Since M is a point on AB such that $AM = MB$, M is called the **mid-point** of AB.

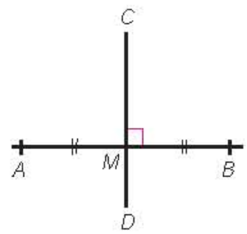


Figure 12.11

Table 12.5 shows the steps in constructing the perpendicular bisector of AB and marking the mid-point of AB.

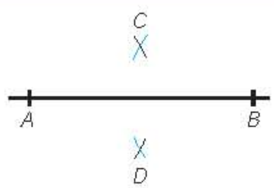
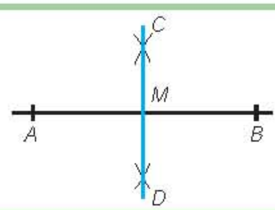
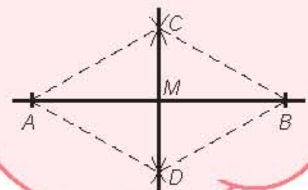
Step 1	With A as the centre, draw an arc with a radius longer than $\frac{1}{2}AB$ on each side of AB.	
Step 2	With B as the centre, draw an arc with a radius same as that in step 1 on each side of AB, which cuts the arcs drawn in step 1 at C and D.	
Step 3	Join CD. CD is the perpendicular bisector of AB. The intersection M of CD and AB is the mid-point of AB.	

Table 12.5

In fact, we can deduce $CD \perp AB$ and $AM = BM$, i.e. CD is the perpendicular bisector of AB, by proving that $\Delta ACD \cong \Delta BCD$ and $\Delta ACM \cong \Delta BCM$.



I.T. Tool

perpendicular bisector 垂直平分綫　　mid-point 中點

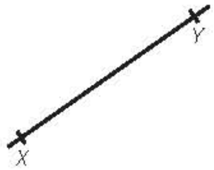
Extension 12.4

Solution

1. Construct the perpendicular bisector of each of the following line segments, and mark the mid-point M of each line segment.

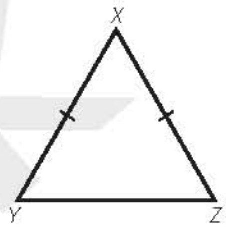
   (a)

   (b)



Solution

2. The figure shows an isosceles triangle XYZ where $XY = XZ$.
   (a) Construct the perpendicular bisector of YZ.
   (b) Does the perpendicular bisector in **(a)** pass through point X?



D Constructing a perpendicular line passing through a point lying outside a line segment

Table 12.6 shows the steps in constructing a line passing through point P and perpendicular to line segment AB.

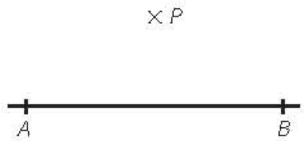


Figure 12.12

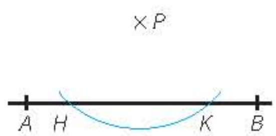
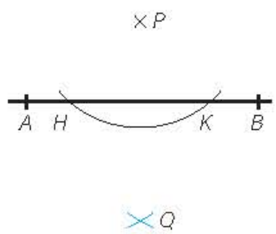
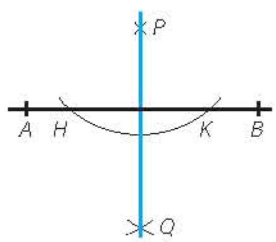
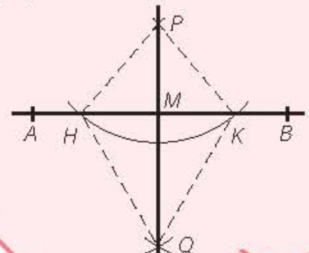
Step 1	With P as the centre, choose an appropriate radius and draw an arc cutting AB at H and K.	
Step 2	With H and K as centres, draw two arcs with radii longer than $\frac{1}{2}HK$, which intersect at Q.	
Step 3	Join PQ, we have $PQ \perp AB$.	

Table 12.6

In fact, we can deduce $PQ \perp AB$ by proving that PQ is the perpendicular bisector of HK.



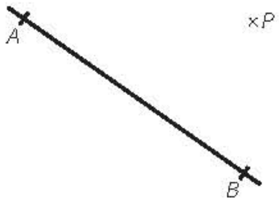
I.T. Tool

Extension 12.5

Solution

1. In each of the following figures, construct a straight line passing through point P and perpendicular to AB.

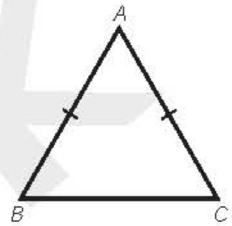
(a)



(b)

Solution

2. The figure shows an isosceles triangle ABC where $AB = AC$.



(a) Construct a straight line passing through point A and perpendicular to BC.

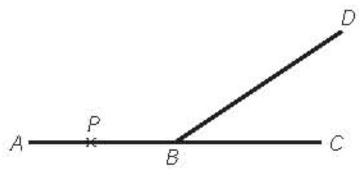
(b) If the straight line constructed in **(a)** meets BC at M, what is the relation between BM and CM?

Skills Upgrading Corner 12.3

[In this exercise, only straight edges and compasses are allowed to be used for construction.]

Solution

1. In the figure, $APBC$ is a straight line.



(a) Construct $\angle QPA$ where $\angle QPA = \angle DBC$ and Q is a point below AC.

(b) What is the relation between PQ and DB? Explain briefly.

Solution

2. In the figure, divide AB into 4 equal parts.

Solution

3. Construct the following angles.

(a) $67.5°$

(b) $210°$

(c) $345°$

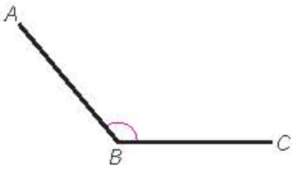
Exercise 12C

[In this exercise, only straight edges and compasses are allowed to be used for construction.]

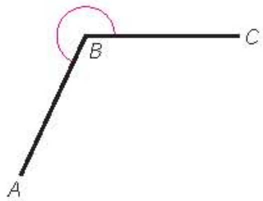
Level 1

1. Copy the following angles.

(a)



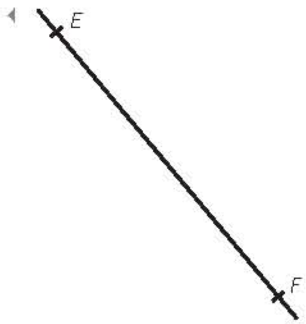
(b)



(c)

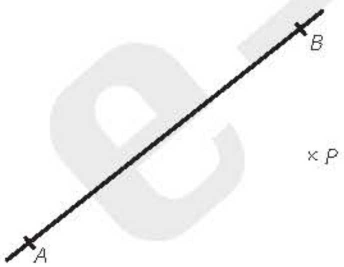
2. (a) Draw an acute angle and then bisect it.
   (b) Draw a reflex angle and then bisect it.
   (c) Draw an obtuse angle and then divide it into four equal angles.

3. Construct the perpendicular bisector of line segment EF.
   [A copy of the figure is provided in the Appendix.]

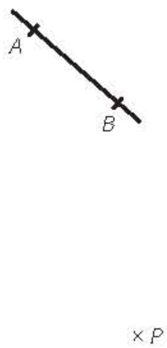


4. In each of the following figures, construct a straight line passing through point P and perpendicular to AB.
   [A copy of each figure is provided in the Appendix.]

(a)



(b)

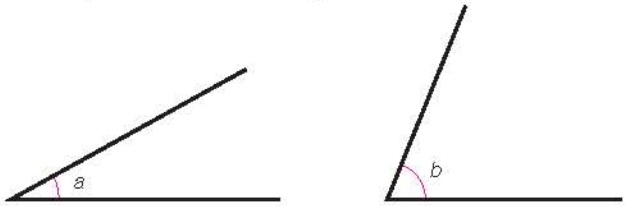


5. Construct the following angles.

(a) $22.5°$	(b) $120°$	(c) $240°$
(d) $165°$	(e) $37.5°$	(f) $157.5°$

Level 2

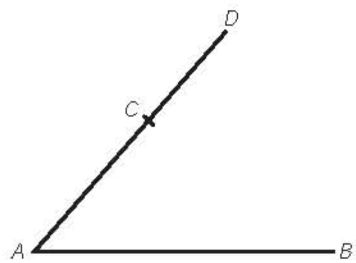
6. The figure shows two angles with sizes of a and b. Construct an angle with the size of $a+b$.



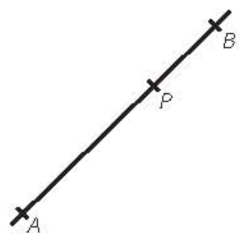
7. The figure shows two angles with sizes of c and d. Construct an angle with the size of $c-d$.

8. In the figure, mark point P on CD such that $CP:PD=3:1$.
[A copy of the figure is provided in the Appendix.]

9. In the figure, ACD is a straight line.
  (a) If E is a point on the right of AD such that $\angle DCE=\angle DAB$, are CE and AB parallel?
  (b) Hence, construct CE.
[A copy of the figure is provided in the Appendix.]



10. In the figure, construct a straight line passing through point P and perpendicular to AB.
[A copy of the figure is provided in the Appendix.]



11. In ΔABC, $AB=6$ cm, $BC=5$ cm and $CA=5$ cm.
  (a) Draw ΔABC. (Use a ruler to obtain the lengths of sides.)
  (b) In the same figure, construct the angle bisector of $\angle ACB$ which intersects AB at D.
  (c) Are ΔACD and ΔBCD congruent? If yes, state the reasons.

12. $\angle A$ is bisected by construction and each of the bisected angle is acute. Can $\angle A$ be a reflex angle? Explain briefly.

12.5 Construction of Regular Polygons

In the previous section, we have learned some basic steps of construction using a straight edge and a pair of compasses. In this section, we will see how regular polygons are constructed without any precise tools in ancient times.

A Constructing equilateral triangles and regular hexagons

Table 12.7 shows the steps in constructing an equilateral triangle and a regular hexagon.

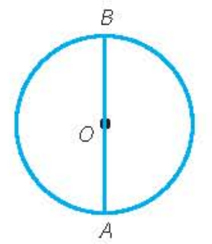
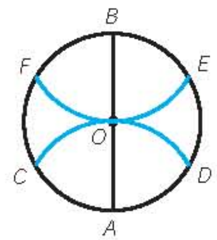
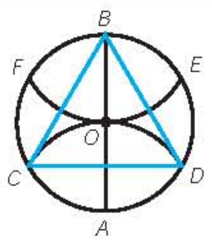
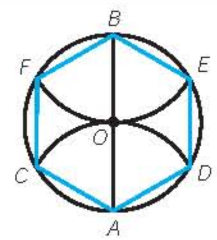
Step 1	Draw a circle using a pair of compasses and keep the distance between the arms of the compasses fixed. Use a straight edge to draw diameter AOB.		Step 2	With A and B as centres, draw two arcs with the radius same as OA, which cut the circumference at C, D, E and F.	
Step 3	Join BC, CD and BD to obtain an equilateral triangle BCD.		Step 4	Join AD, DE, EB, BF, FC and CA to obtain a regular hexagon $ADEBFC$.	

Table 12.7

I.T. Tool

Extension 12.6

Solution

(a) Draw a circle with O as the centre and OA as the radius. Construct an equilateral triangle inside the circle and measure each interior angle.

× O

× A

(b) Draw a circle with O as the centre and OB as the radius. Construct a regular hexagon inside the circle and measure each interior angle.

× O

× B

B Constructing regular octagons

Table 12.8 shows the steps in constructing a regular octagon.

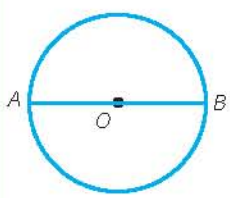
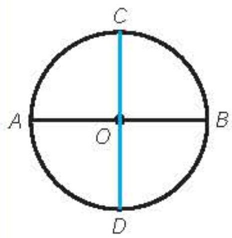
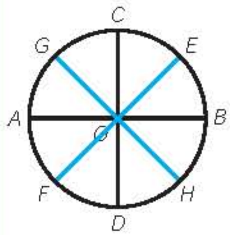
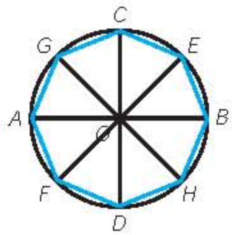
Step 1	Draw a circle using a pair of compasses and draw diameter AOB using a straight edge.		Step 2	Construct the perpendicular bisector of AB, which intersects the circumference at C and D. COD is another diameter of the circle.	
Step 3	Construct the angle bisectors of $\angle BOC$ and $\angle AOC$, which intersect the circumference at E, F, G and H.		Step 4	Join AF, FD, DH, HB, BE, EC, CG and GA to obtain a regular octagon $AFDHBECG$.	

Table 12.8

Student's CD

I.T. Tool

Notes:
(a) Refer to Table 12.5 (page 12.21) for the steps in constructing a perpendicular bisector.
(b) Refer to Table 12.2 (page 12.18) for the steps in constructing an angle bisector.
(c) Joining AD, DB, BC and CA will obtain a square $ADBC$.

Extension 12.7

Solution

Draw a circle with O as the centre and OA as the radius. Construct a regular octagon inside the circle and measure each interior angle.

× O

× A

C Constructing regular pentagons

Table 12.9 shows the steps in constructing a regular pentagon.

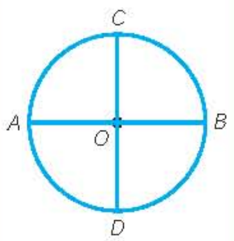
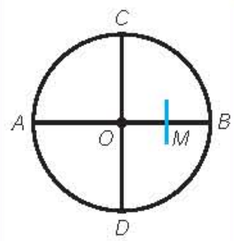
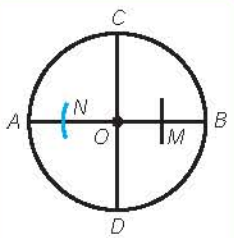
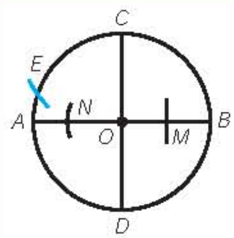
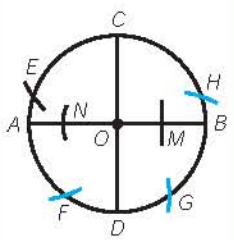
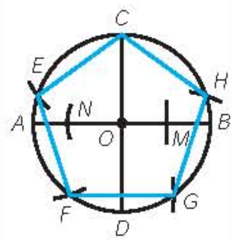
Step 1	Draw a circle and construct diameters AOB and COD perpendicular to each other.		Step 2	Construct the perpendicular bisector of OB to obtain mid-point M.	
Step 3	With M as the centre and CM as the radius, draw an arc cutting AO at N.		Step 4	With C as the centre and CN as the radius, draw an arc cutting the circumference at E. Keep the distance between the arms of the compasses fixed.	
Step 5	With E as the centre, draw an arc with the radius same as that in step 4, which cuts the circumference at F, and then mark G and H in a similar way.		Step 6	Join CE, EF, FG, GH and HC to obtain a regular pentagon $CEFGH$.	

Table 12.9

I.T. Tool

Extension 12.8

Solution

Draw a circle with O as the centre and OA as the radius. Construct a regular pentagon inside the circle and measure each interior angle.

× O

× A

At this stage, we have only introduced some simple constructions of regular polygons for appreciation. In fact, these constructions involve many geometric theories behind and each of them may not be the only construction for each polygon mentioned.

Chapter Summary

Chapter Summary

A. Term Introduced

[This is a quiz to check your understanding of some special terms in this chapter. Match items in column A to column B appropriately.]

Column A			Column B
1. Tessellation	•	•	**(a)** Repeated use of regular polygons of one kind in the same size to cover a plane without gaps and without overlaps.
2. Regular tessellation	•	•	**(b)** Repeated use of figures of one kind or more to cover a plane without gaps and without overlaps.
3. Angle bisector	•	•	**(c)** A straight line which divides a line segment into two equal parts and is perpendicular to this line segment.
4. Perpendicular bisector	•	•	**(d)** A straight line which divides an angle into two equal halves.

B. Fact to Remember

1. The sum of all interior angles of an n-gon is $(n-2)\times 180°$.
   [Abbreviation: $\angle$ sum of polygon]

2. The sum of all exterior angles of a convex polygon is 360°.
   [Abbreviation: sum of ext. $\angle$s of polygon]

3. For regular polygons, only equilateral triangles, squares and regular hexagons can tessellate.

Non-foundation Part

4. Using a straight edge and a pair of compasses, we can
   (a) copy an angle.
   (b) construct an angle bisector.
   (c) construct a perpendicular bisector.
   (d) construct a perpendicular line passing through a point outside a line segment.

End

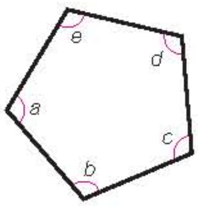
Check Yourself

[This is a quiz to remind you of the basic concepts you have learned in this chapter. Each question tests a concept under the section listed on the right. Failure in any part of a question indicates a need to do a revision on the section listed.]

Section

1. **(a)** In the figure, — 12.1

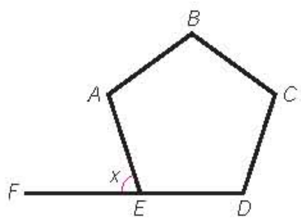
$a+b+c+d+e=$ ______________ .



(b) Find the size of an interior angle of a regular nonagon.

2. **(a)** In the figure, $ABCDE$ is a regular pentagon and FED is a straight line. — 12.2

$x=$ ______________



(b) Find the size of an exterior angle of a regular octagon.

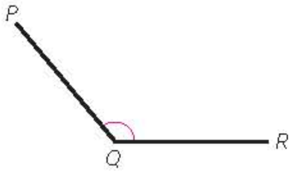
3. **(a)** State the names of all kinds of regular polygons which can tessellate. 12.3B

(b) What characteristic do their interior angles have?

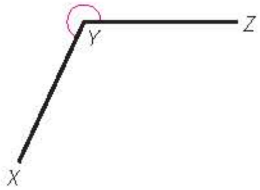
Non-foundation Part

4. Copy the following angles using a straight edge and a pair of compasses. 12.4A

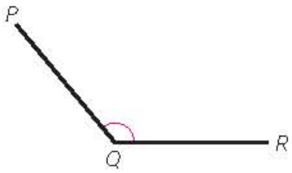
(a)



(b)



5. **(a)** Construct the angle bisector of $\angle PQR$ in the following figure. 12.4B



(b) Construct an angle of 15° using a straight edge and a pair of compasses.

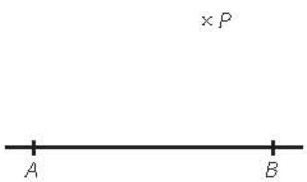
6. Construct the perpendicular bisector of each of the following line segments. 12.4C

(a)

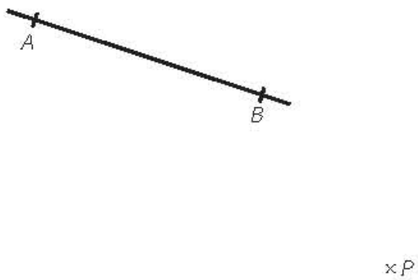
(b)

7. In each of the following figures, construct a straight line passing through point P and perpendicular to AB. 12.4D

(a)



(b)



End

Revision Exercise 12

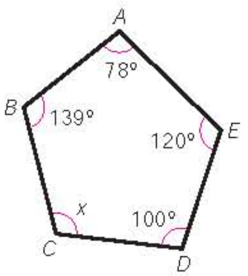
Level 1

1. Find the size of an exterior angle of a regular 15-gon.

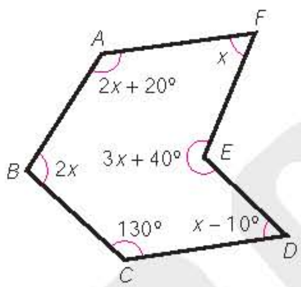
2. Find the size of an interior angle of a regular 24-gon.

3. Find x in each of the following figures.

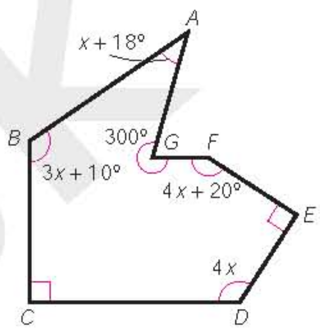
(a)



(b)

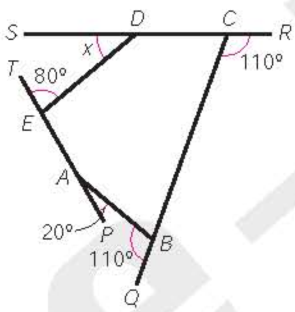


(c)

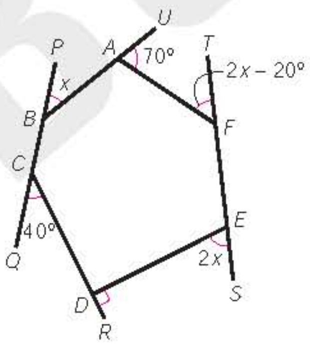


4. In each of the following figures, the marked angles are exterior angles of the polygon. Find x.

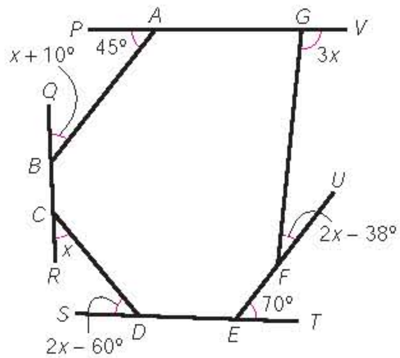
(a)



(b)



(c)

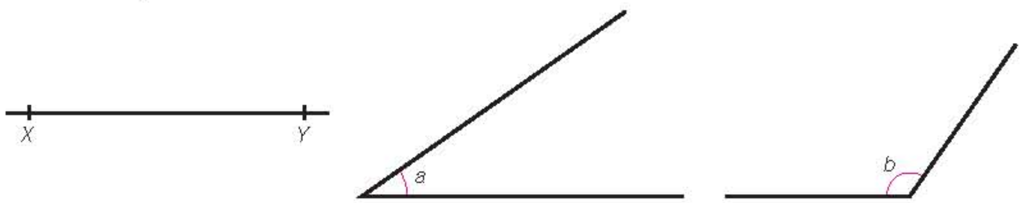


Non-foundation Part

5. Construct the following angles using a straight edge and a pair of compasses.

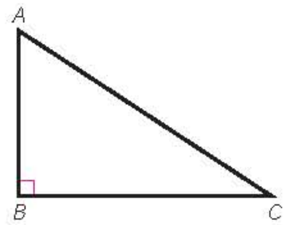
(a) 52.5° **(b)** 112.5° **(c)** 292.5°

6. The figure shows a line segment XY and two angles with sizes of a and b. If $AB = XY$, $\angle BAC = a$ and $\angle ABC = b$, construct ΔABC.



7. The figure shows a right-angled triangle ABC where $\angle ABC = 90°$.
   (a) Construct the perpendicular bisector of each side of ΔABC.
   (b) Do the perpendicular bisectors in **(a)** intersect at the same point? If yes, what characteristic does the position of this intersecting point have ?

   [A copy of the figure is provided in the Appendix.]

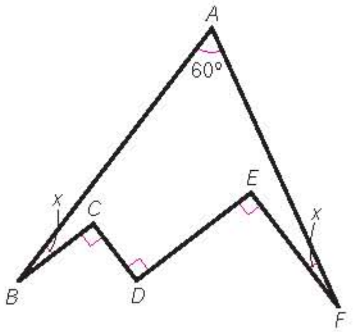


End

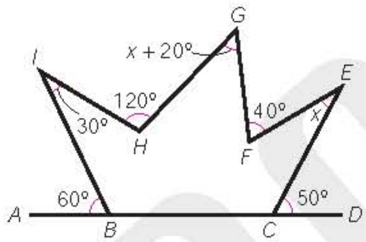
Level 2

8. Find x in each of the following figures.

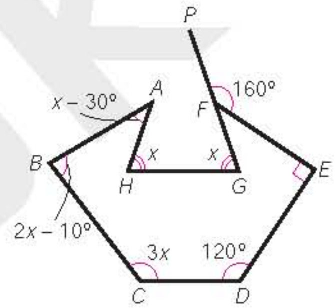
   (a)



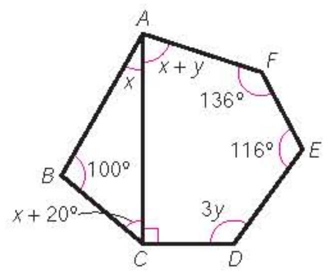
   (b) $ABCD$ is a straight line.



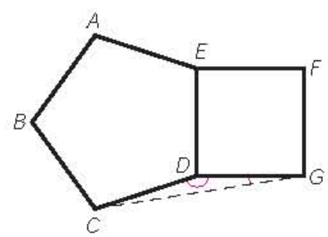
   (c) PFG is a straight line.



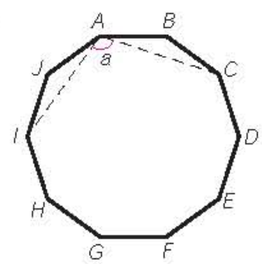
9. Find x and y in the figure.



10. In the figure, $ABCDE$ is a regular pentagon and $DEFG$ is a square. Find $\angle CDG$ and $\angle DGC$.



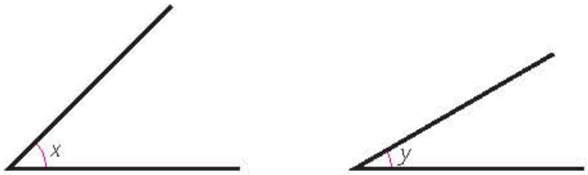
11. In the figure, $ABCDEFGHIJ$ is a regular decagon. Find a.



12. If 5 times an exterior angle of a regular polygon is 60° less than its interior angle, find the number of sides of the polygon.

Non-foundation Part

13. The figure shows two angles with sizes of x and y.



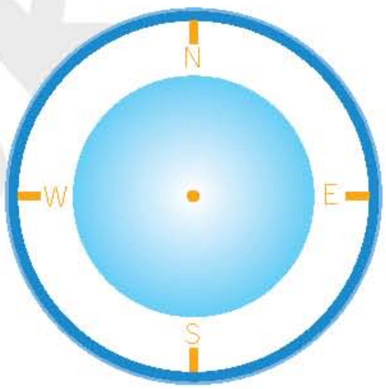
Construct angles with the following sizes.

(a) $x + y$

(b) $x - y$

(c) $3y$

14. The figure shows a compass. Draw a needle pointing to N22.5°E using a straight edge and a pair of compasses.
[A copy of the figure is provided in the Appendix.]



15. In the figure, the respective lengths of the line segments are a and b. Construct two different triangles with base a and height b.

16. **(a)** Draw a triangle, and mark the mid-point of each side by construction.

(b) Join each mid-point to the vertex opposite to it.

(c) Do the three lines intersect at the same point?

17. **(a)** Draw a circle and mark three points A, B and C on the circumference. Then join AB, BC and CA to form ΔABC.

(b) Construct the angle bisector of $\angle ABC$.

(c) Construct the perpendicular bisector of AC.

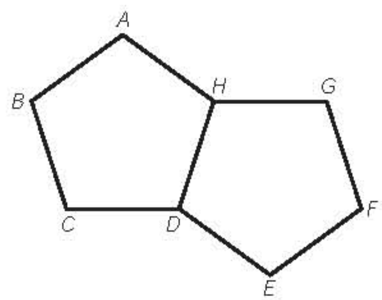
(d) Do the lines constructed in **(b)** and **(c)** intersect on the circumference?

End

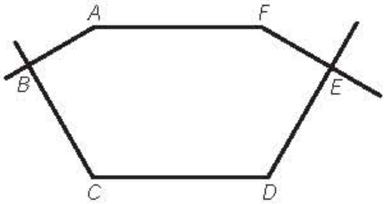
18. In the figure, $ABCDH$ and $HDEFG$ are two regular pentagons.

(a) Find $\angle AHG$.

(b) If $\angle AHG$ is equal to the size of an interior angle of a regular polygon, find the number of sides of this regular polygon.



19. In the figure, a hexagon $ABCDEF$ is formed by partially overlapping a regular n-gon and a regular $2n$-gon, where $\angle BCD$ and $\angle CDE$ are equal to the size of an interior angle of the regular n-gon, and $\angle BAF$ and $\angle AFE$ are equal to the size of an interior angle of the regular $2n$-gon.

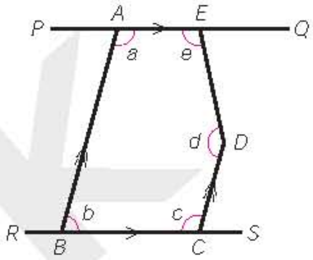


(a) Express $\angle BCD$ in terms of n.

(b) Express $\angle BAF$ in terms of n.

(c) If $\angle ABC = \angle FED = 90°$, find the value of n.

20. In the figure, $PAEQ$ and $RBCS$ are straight lines, $PQ \parallel RS$ and $AB \parallel DC$.

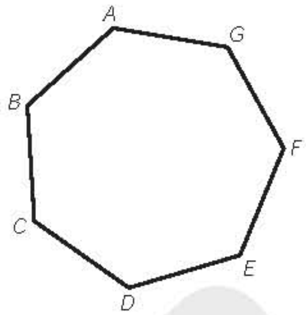


(a) Prove that $a = c$.

(b) Prove that $a + d + e = 360°$.

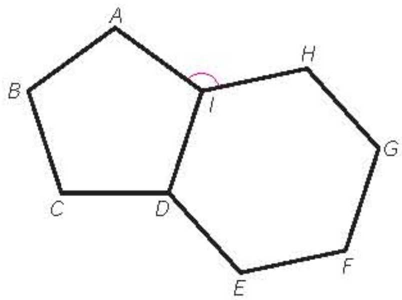
MC Question

21. In the figure, the sum of all interior angles of polygon $ABCDEFG$ is



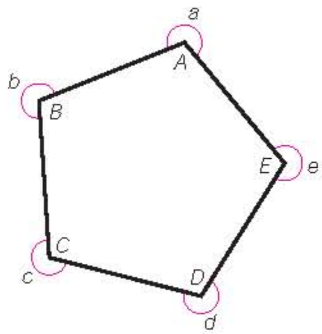
A. 900°.
B. 1 080°.
C. 1 260°.
D. 1 800°.

22. In the figure, $ABCDI$ is a regular pentagon and $DEFGHI$ is a regular hexagon. Find $\angle AIH$.



A. 120°
B. 132°
C. 144°
D. 150°

23. In the figure, $a + b + c + d + e =$



A. 360°.
B. 900°.
C. 1 260°.
D. 1 800°.

24. If an interior angle of a regular polygon is twice its exterior angle, then this polygon is

A. an equilateral triangle.
B. a square.
C. a regular pentagon.
D. a regular hexagon.

25. Which of the following must not be an exterior angle of a regular polygon?

A. 45°
B. 60°
C. 75°
D. 90°

26. Which of the following is/are true?

I. All regular polygons can tessellate.
II. All polygons which can tessellate are regular polygons.

A. I only
B. II only
C. I and II
D. None of the above

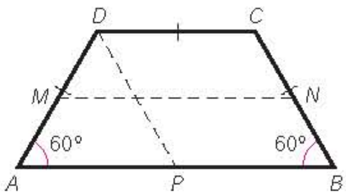
Problem-solving and Exploring

Hint for the Title Page Question

(a) Construct a line segment passing through D and parallel to BC such that the line segment meets AB at P.

(b) Mark the mid-points of AD and BC as M and N respectively. Join MN.

(c) Construct some additional lines in a similar way to divide trapezium $ABCD$ into four identical trapeziums.

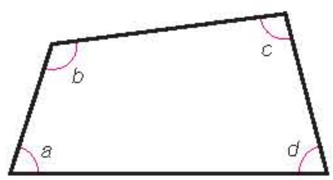


Additional Question

1. (a) Explain why any kind of parallelograms can tessellate.

(b) Explain why any kind of triangles can tessellate.

2. (a) The figure shows a quadrilateral. By using the geometric knowledge acquired in this chapter, explain why angles a, b, c and d form a round angle.



(b) Combine four quadrilaterals identical to the one on the right, show that angles a, b, c and d form a round angle.
[A copy of the figure is provided in the Appendix.]

(c) Draw 10 identical convex quadrilaterals on a piece of paper. Then cut them out to tessellate a plane.

(d) Draw 10 identical concave quadrilaterals on a piece of paper. Then cut them out to tessellate a plane.

Appreciating Mathematics — Enrichment

Creating Tessellated Patterns

With some imagination, figures which can tessellate can be used to create beautiful tessellated patterns after some basic transformations (e.g. translation, rotation and reflection). Here are some examples:

A Translation

Translation refers to the process of moving a figure along a certain direction. For example:

(a)

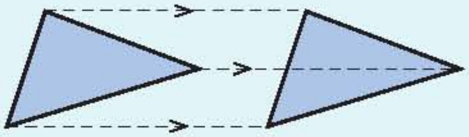


Figure 12.13

(b)

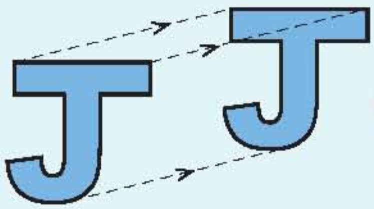


Figure 12.14

Table 12.10 shows an example of creating tessellated patterns by using squares and translation.

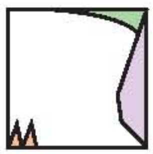
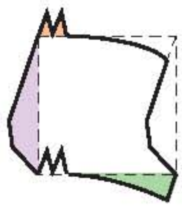
Step 1	Design some patterns along the sides of a square.		Step 2	Translate the pattern on each side of the square in step 1 to the opposite side.	
Step 3	A new figure is obtained.		Step 4	Use the new figure for tessellation.	

Table 12.10

Student's CD

I.T. Tool

B Rotation

Rotation refers to the process of turning a figure about a fixed point through a certain angle. For example:

(a)

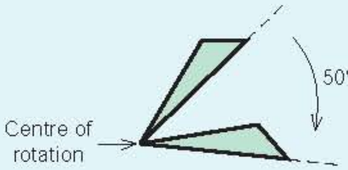


Figure 12.15

(b)

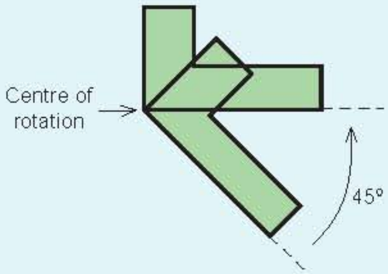


Figure 12.16

Table 12.11 shows an example of creating tessellated patterns by using equilateral triangles and rotation.

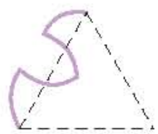
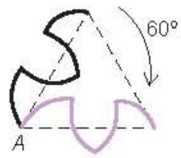
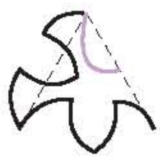
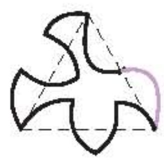
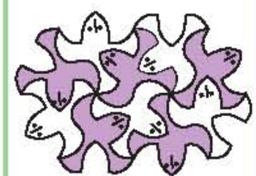
Step 1	Imagine there is an equilateral triangle. On one of its sides, design a pattern of curves.		Step 2	Take A as the centre of rotation, rotate the pattern of curves clockwise through 60°.	
Step 3	Divide the side without any curves into two equal halves. Design a new pattern of curves on the upper half.		Step 4	Take the mid-point of the side mentioned in step 3 as the centre of rotation, rotate the new pattern of curves in step 3 through 180°.	
Step 5	A new figure is obtained.		Step 6	Use the new figure for tessellation.	

Table 12.11

I.T. Tool

C Reflection

If a figure is reflected along a straight line, the result looks like the mirror image of the figure. For example:

(a)

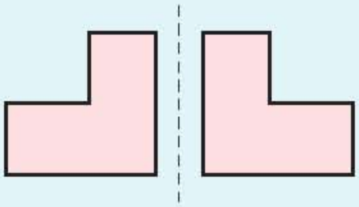


Figure 12.17

(b)

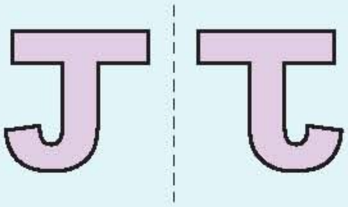


Figure 12.18

Table 12.12 shows an example of creating tessellated patterns by using squares and reflection.

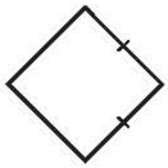
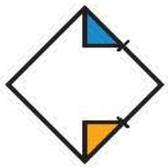
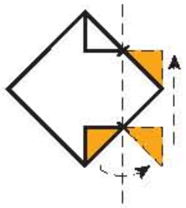
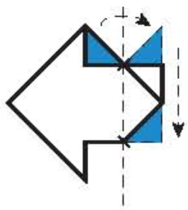
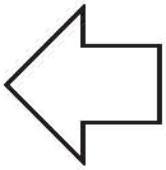
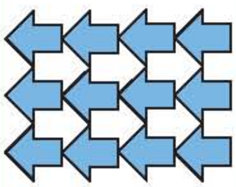
Step 1	Mark the mid-points of two adjacent sides of a square.		Step 2	Design a pattern along each of the adjacent sides in step 1.	
Step 3	Reflect the lower shaded pattern along the dotted line and translate it upwards.		Step 4	Reflect the upper shaded pattern along the dotted line and translate it downwards.	
Step 5	A new figure is obtained.		Step 6	Use the new figure for tessellation.	

Table 12.12

I.T. Tool

Using the above methods, different tessellated patterns can be easily designed.

The following are some works of Escher (1898 - 1972), a famous Dutch artist, involving tessellated patterns.

(a) (b) (c)

Figure 12.19

Extension 12.9 Solution

Design tessellated patterns by applying the techniques learned in this section.